Solved Paper 2022

ONLINE

25th July 2nd Shift

MATHEMATICS

SECTION-A (MULTIPLE CHOICE QUESTIONS)

1. For $z \in C$ if the minimum value of $\left(|z-3\sqrt{2}|+|z-p\sqrt{2}i|\right)$ is $5\sqrt{2}$, then a value of p is _____.

(a) 3 (b) $\frac{7}{2}$ (c) 4 (d) $\frac{9}{2}$

2. The number of real values of λ, such that the system of linear equations
$2x - 3y + 5z = 9$, $x + 3y - z = -18$, $3x - y + (\lambda^2 - |\lambda|)z = 16$ has no solutions, is

(a) 0 (b) 1 (c) 2 (d) 4

3. The number of bijective functions
$f: \{1, 3, 5, 7, ..., 99\} \to \{2, 4, 6, 8, ..., 100\}$, such that
$f(3) \geq f(9) \geq f(15) \geq f(21) \geq ... \geq f(99)$, is ______.

(a) ${}^{50}P_{17}$ (b) ${}^{50}P_{33}$

(c) $33! \times 17!$ (d) $\frac{50!}{2}$

4. The remainder when $(11)^{1011} + (1011)^{11}$ is divided by 9 is

(a) 1 (b) 4 (c) 6 (d) 8

5. The sum $\sum_{n=1}^{21} \frac{3}{(4n-1)(4n+3)}$ is equal to

(a) $\frac{7}{87}$ (b) $\frac{7}{29}$ (c) $\frac{14}{87}$ (d) $\frac{21}{29}$

6. $\lim_{x \to \frac{\pi}{4}} \frac{8\sqrt{2}-(\cos x+\sin x)^7}{\sqrt{2}-\sqrt{2}\sin 2x}$ is equal to

(a) 14 (b) 7 (c) $14\sqrt{2}$ (d) $7\sqrt{2}$

7. $\lim_{n \to \infty} \frac{1}{2^n}\left(\frac{1}{\sqrt{1-\frac{1}{2^n}}}+\frac{1}{\sqrt{1-\frac{2}{2^n}}}+\frac{1}{\sqrt{1-\frac{3}{2^n}}}+...+\frac{1}{\sqrt{1-\frac{2^n-1}{2^n}}}\right)$ is equal to

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) -2

8. If A and B are two events such that $P(A)=\frac{1}{3}, P(B)=\frac{1}{5}$ and $P(A \cup B)=\frac{1}{2}$, then $P(A|B') + P(B|A')$ is equal to

(a) $\frac{3}{4}$ (b) $\frac{5}{8}$ (c) $\frac{5}{4}$ (d) $\frac{7}{8}$

9. Let $[t]$ denote the greatest integer less than or equal to t. Then the value of the integral $\int_{-3}^{101}\left([\sin(\pi x)]+e^{[\cos(2\pi x)]}\right)dx$ is equal to

(a) $\frac{52(1-e)}{e}$ (b) $\frac{52}{e}$

(c) $\frac{52(2+e)}{e}$ (d) $\frac{104}{e}$

10. Let the point $P(\alpha, \beta)$ be at a unit distance from each of the two lines $L_1 : 3x - 4y + 12 = 0$, and $L_2 : 8x + 6y + 11 = 0$. If P lies below L_1 and above L_2, then $100(\alpha + \beta)$ is equal to

(a) -14 (b) 42 (c) -22 (d) 14

11. Let a smooth curve $y = f(x)$ be such that the slope of the tangent at any point (x, y) on it is directly proportional to $\left(\frac{-y}{x}\right)$. If the curve passes through the points (1, 2) and (8, 1), then $\left|y\left(\frac{1}{8}\right)\right|$ is equal to

(a) $2\log_e 2$ (b) 4 (c) 1 (d) $4\log_e 2$

12. If the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the line $\frac{x}{7}+\frac{y}{2\sqrt{6}}=1$ on the x-axis and the line $\frac{x}{7}-\frac{y}{2\sqrt{6}}=1$ on the y-axis, then the eccentricity of the ellipse is

(a) $\frac{5}{7}$ (b) $\frac{2\sqrt{6}}{7}$ (c) $\frac{3}{7}$ (d) $\frac{2\sqrt{5}}{7}$

13. The tangents at the points $A(1, 3)$ and $B(1, -1)$ on the parabola $y^2 - 2x - 2y = 1$ meet at the point P. Then the area (in unit^2) of the triangle PAB is :

(a) 4 (b) 6 (c) 7 (d) 8

14. Let the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{7}=1$ and the hyperbola $\frac{x^2}{144}-\frac{y^2}{\alpha}=\frac{1}{25}$ coincide. Then the length of the latus rectum of the hyperbola is :

(a) $\frac{32}{9}$ (b) $\frac{18}{5}$ (c) $\frac{27}{4}$ (d) $\frac{27}{10}$

15. A plane E is perpendicular to the two planes $2x - 2y + z = 0$ and $x - y + 2z = 4$, and passes through the point $P(1, -1, 1)$. If the distance of the plane E from the point $Q(a, a, 2)$ is $3\sqrt{2}$, then $(PQ)^2$ is equal to

(a) 9 (b) 12 (c) 21 (d) 33

16. The shortest distance between the lines $\frac{x+7}{-6}=\frac{y-6}{7}=z$ and $\frac{7-x}{2}=y-2=z-6$ is

(a) $2\sqrt{29}$ (b) 1 (c) $\sqrt{\frac{37}{29}}$ (d) $\frac{\sqrt{29}}{2}$

17. Let $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$ and let $\vec{b}$ be a vector such that $\vec{a} \times \vec{b} = 2\hat{i} - \hat{k}$ and $\vec{a} \cdot \vec{b} = 3$. Then the projection of $\vec{b}$ on the vector $\vec{a} - \vec{b}$ is :

(a) $\frac{2}{\sqrt{21}}$ (b) $2\sqrt{\frac{3}{7}}$

(c) $\frac{2}{3}\sqrt{\frac{7}{3}}$ (d) $\frac{2}{3}$

18. If the mean deviation about median for the numbers 3, 5, 7, $2k$, 12, 16, 21, 24, arranged in the ascending order, is 6 then the median is

(a) 11.5 (b) 10.5 (c) 12 (d) 11

19. $2\sin\left(\frac{\pi}{22}\right)\sin\left(\frac{3\pi}{22}\right)\sin\left(\frac{5\pi}{22}\right)\sin\left(\frac{7\pi}{22}\right)\sin\left(\frac{9\pi}{22}\right)$ is equal to :

(a) $\frac{3}{16}$ (b) $\frac{1}{16}$ (c) $\frac{1}{32}$ (d) $\frac{9}{32}$

20. Consider the following statements :

P : Ramu is intelligent.
Q : Ramu is rich.
R : Ramu is not honest.

The negation of the statement "Ramu is intelligent and honest if and only if Ramu is not rich" can be expressed as :

(a) $((P \wedge (\sim R)) \wedge Q) \wedge ((\sim Q) \wedge ((\sim P) \vee R))$
(b) $((P \wedge R) \wedge Q) \vee ((\sim Q) \wedge ((\sim P) \vee (\sim R)))$
(c) $((P \wedge R) \wedge Q) \wedge ((\sim Q) \wedge ((\sim P) \vee (\sim R)))$
(d) $((P \wedge (\sim R)) \wedge Q) \vee ((\sim Q) \wedge ((\sim P) \vee R))$

SECTION - B (NUMERICAL VALUE TYPE)

Attempt any 5 questions out of 10.

21. Let $A = \{1, 2, 3, 4, 5, 6, 7\}$. Define $B = \{T \subseteq A$: either $1 \notin T$ or $2 \in T\}$ and $C = \{T \subseteq A : T$ the sum of all the elements of T is a prime number$\}$. Then the number of elements in the set $B \cup C$ is ________.

22. Let $f(x)$ be a quadratic polynomial with leading coefficient 1 such that $f(0) = p$, $p \neq 0$, and $f(1) = \frac{1}{3}$. If the equations $f(x) = 0$ and $fofofof(x) = 0$ have a common real root, then $f(-3)$ is equal to ________.

23. Let $A = \begin{bmatrix} 1 & a & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$, $a, b \in R$. If for some $n \in N$,

$A^n = \begin{bmatrix} 1 & 48 & 2160 \\ 0 & 1 & 96 \\ 0 & 0 & 1 \end{bmatrix}$ then $n + a + b$ is equal to ______.

24. The sum of the maximum and minimum values of the function $f(x) = |5x - 7| + [x^2 + 2x]$ in the interval $\left[\frac{5}{4}, 2\right]$, where $[t]$ is the greatest integer $\leq t$, is ________.

25. Let $y = y(x)$ be the solution of the differential equation $\frac{dy}{dx} = \frac{4y^3 + 2yx^2}{3xy^2 + x^3}$, $y(1) = 1$.

If for some $n \in N$, $y(2) \in [n - 1, n)$, then n is equal to ______.

26. Let f be a twice differentiable function on R. If $f'(0) = 4$ and $f(x) + \int_0^x (x - t) f'(t) dt = \left(e^{2x} + e^{-2x}\right)\cos 2x + \frac{2}{a}x$, then $(2a + 1)^5 a^2$ is equal to ________.

27. Let $a_n = \int_{-1}^{n} \left(1 + \frac{x}{2} + \frac{x^2}{3} + \ldots + \frac{x^{n-1}}{n}\right) dx$ for every $n \in N$. Then the sum of all the elements of the set $\{n \in N : a_n \in (2, 30)\}$ is ________.

28. If the circles $x^2 + y^2 + 6x + 8y + 16 = 0$ and $x^2 + y^2 + 2(3 - \sqrt{3})x + 2(4 - \sqrt{6})y = k + 6\sqrt{3} + 8\sqrt{6}$, $k > 0$, touch internally at the point $P(\alpha, \beta)$, then $(\alpha + \sqrt{3})^2 + (\beta + \sqrt{6})^2$ is equal to ______.

29. Let the area enclosed by the x-axis, and the tangent and normal drawn to the curve $4x^3 - 3xy^2 + 6x^2 - 5xy - 8y^2 + 9x + 14 = 0$ at the point $(-2, 3)$ be A. Then $8A$ is equal to ________.

30. Let $x = \sin(2\tan^{-1}\alpha)$ and $y = \sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$.

If $S = \{\alpha \in R : y^2 = 1 - x\}$, then $\sum_{\alpha \in S} 16\alpha^3$ is equal to

HINTS & EXPLANATIONS

1. **(c) :** We know that, $|z_1 - z_2| \leq |z_1| + |z_2|$

$\Rightarrow |(z - 3\sqrt{2}) - (z - p\sqrt{2}i)| \leq |z - 3\sqrt{2}| + |z - p\sqrt{2}i|$

$\Rightarrow |-3\sqrt{2} + p\sqrt{2}i| \leq \left(|z - 3\sqrt{2}| + |z - p\sqrt{2}i|\right)$

$\Rightarrow \sqrt{18 + 2p^2} \leq \left(|z - 3\sqrt{2}| + |z - p\sqrt{2}i|\right)$

Given, $\sqrt{18 + 2p^2} = 5\sqrt{2}$

$\Rightarrow 18 + 2p^2 = 50 \Rightarrow p^2 = 16 \Rightarrow p = \pm 4$

2. **(c) :** The system of equations has no solution when $\Delta = 0$

$$\begin{vmatrix} 2 & -3 & 5 \\ 1 & 3 & -1 \\ 3 & -1 & (\lambda^2 - |\lambda|) \end{vmatrix} = 0$$

Expanding along R_1, we get

$2[3(\lambda^2 - |\lambda|) - 1] + 3[\lambda^2 - |\lambda|) + 3] + 5(-1 - 9) = 0$

$\Rightarrow \; 6\lambda^2 - 6|\lambda| - 2 + 3\lambda^2 - 3|\lambda| + 9 - 50 = 0$

$\Rightarrow \; 9\lambda^2 - 9|\lambda| - 43 = 0$

When $\lambda < 0$ then, $9\lambda^2 + 9\lambda - 43 = 0$

$\Rightarrow \; \lambda = \dfrac{-9 \pm \sqrt{1629}}{18}$

When $\lambda > 0$ then, $9\lambda^2 - 9\lambda - 43 = 0$

$\lambda = \dfrac{9 \pm \sqrt{1629}}{18}$

So, the real values are ±2.74 satisfy the given condition.

Hence, number of real values of λ is 2.

3. (b) : $f : \{1, 3, 5, 7, ..., 99\} \to \{2, 4, 6, 8, ..., 100\}$

Let $f : A \to B$ where $A = \{1, 3, 5, 7, ..., 99\}$ and $B = \{2, 4, 6, ..., 100\}$

As number of elements in A and B is 50.

$\therefore$ Total possibilities = 50

Such that $f(3) \ge f(9) \ge f(15) \ge f(21) \ge .... \ge f(99)$

We have, A.P. = 3, 9, 15, 21, ..., 99

Here, $a_n = 99$, $a_1 = 3$ and $d = 6$

$a_n = a + (n-1)d$

$\Rightarrow \; 99 = 3 + (n-1)6 \Rightarrow n = 17$

Hence, number of bijective function = ${}^{50}C_{17} \times 33! = {}^{50}P_{33}$

4. (d) : $(11)^{1011} + (1011)^{11} = (9+2)^{1011} + (1008+3)^{11}$

When this is divided by 9, we are left with

$2^{1011} + 3^{11} = (8)^{337} + 3^{11} = (9-1)^{337} + 3^{11}$ = multiple of 9 $-1 + 9^5 \cdot 3$

Remainder = $-1 + 9 = 8$

5. (b) : We have, $\displaystyle\sum_{n=1}^{21} \frac{3}{(4n-1)(4n+3)}$

$$= \frac{3}{4}\sum_{n=1}^{21}\left[\frac{1}{4n-1} - \frac{1}{4n+3}\right]$$

$$= \frac{3}{4}\left[\left(\frac{1}{3}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{11}\right)+\left(\frac{1}{11}-\frac{1}{15}\right)+...\right] = \frac{3}{4}\left[\frac{1}{3}-\frac{1}{87}\right] = \frac{7}{29}$$

6. (a) : $\displaystyle\lim_{x \to \frac{\pi}{4}} \frac{8\sqrt{2} - (\cos x + \sin x)^7}{\sqrt{2} - \sqrt{2}\sin 2x}$

The given limit is of the form $\frac{0}{0}$ so by applying L'Hospital's rule we get,

$$\lim_{x \to \frac{\pi}{4}} \frac{0 - 7(\cos x + \sin x)^6(-\sin x + \cos x)}{0 - 2\sqrt{2}\cos 2x}$$

Again, it is of the form $\frac{0}{0}$, so applying L'Hospital's rule, we get

$$\lim_{x \to \frac{\pi}{4}} \frac{-7[(\cos x + \sin x)^6(-\cos x - \sin x) + (-\sin x + \cos x)\, 6(\cos x + \sin x)^5(-\sin x + \cos x)]}{4\sqrt{2}\sin 2x}$$

$$= \frac{-7\left[(\sqrt{2})^6(-\sqrt{2}) + 0\right]}{4\sqrt{2}} = 14$$

7. (c) : $\displaystyle\lim_{n \to \infty} \frac{1}{2^n}\left(\frac{1}{\sqrt{1-\frac{1}{2^n}}} + \frac{1}{\sqrt{1-\frac{2}{2^n}}} + \frac{1}{\sqrt{1-\frac{3}{2^n}}} + ... + \frac{1}{\sqrt{1-\frac{2^n-1}{2^n}}}\right)$

$$= \lim_{n \to \infty} \frac{1}{2^n}\sum_{x=1}^{2^n-1} \frac{1}{\sqrt{1-\frac{x}{2^n}}}$$

Sum of series = $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x}}\, dx$

$= \left[-2\sqrt{1-x}\right]_0^1 = [-2(0) + 2(1)] = 2$

8. (b) : Given $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{5}$ $P(A \cup B) = \frac{1}{2}$

$P(A \cap B) = \frac{1}{3} + \frac{1}{5} - \frac{1}{2} = \frac{1}{30}$

Now, $P(A|B') + P(B|A') = \dfrac{P(A \cap B')}{P(B')} + \dfrac{P(B \cap A')}{P(A')}$

$$= \frac{P(A) - P(A \cap B)}{P(B')} + \frac{P(B) - P(A \cap B)}{P(A')}$$

$$= \frac{\frac{1}{3} - \frac{1}{30}}{\frac{4}{5}} + \frac{\frac{1}{5} - \frac{1}{30}}{\frac{2}{3}} = \frac{3}{8} + \frac{1}{4} = \frac{5}{8}$$

9. (b) : Let $I = \displaystyle\int_{-3}^{101} \left([\sin(\pi x)] + e^{[\cos(2\pi x)]}\right) dx$

$$= \int_{-3}^{101} [\sin(\pi x)]\, dx + \int_{-3}^{101} e^{[\cos(2\pi x)]}\, dx$$

$$= 52\int_0^2 [\sin(\pi x)]\, dx + 104\int_0^1 e^{[\cos(2\pi x)]}\, dx$$

Let $I_1 = \displaystyle\int_0^2 [\sin(\pi x)]\, dx$ and $I_2 = \displaystyle\int_0^1 e^{[\cos(2\pi x)]}\, dx$

$$I_1 = \int_0^1 [\sin(\pi x)]\, dx + \int_1^2 [\sin(\pi x)]\, dx = 0 + \int_1^2 (-1)\, dx = -1$$

$$I_2 = \int_0^{1/4} e^{[\cos(2\pi x)]}\, dx + \int_{1/4}^{3/4} e^{[\cos(2\pi x)]}\, dx + \int_{3/4}^{1} e^{[\cos(2\pi x)]}\, dx$$

$$= \int_0^{1/4} e^0\, dx + \int_{1/4}^{3/4} e^{-1}\, dx + \int_{3/4}^{1} e^0\, dx$$

$$= \frac{1}{4} + \frac{1}{e}\left[\frac{3}{4} - \frac{1}{4}\right] + \left(1 - \frac{3}{4}\right) = \frac{1}{2} + \frac{1}{e}\left(\frac{1}{2}\right)$$

$$\therefore \; I = 52(-1) + \frac{104}{2}\left(1 + \frac{1}{e}\right) = -52 + 52 + \frac{52}{e} = \frac{52}{e}$$

10. (d) : Given, lines $L_1 : 3x - 4y + 12 = 0$ and $L_2 : 8x + 6y + 11 = 0$ and P lies below L_1 and above L_2.

$\therefore\ L_1(\alpha, \beta) > 0$

Now, $\left|\frac{3\alpha - 4\beta + 12}{5}\right| = 1 \Rightarrow 3\alpha - 4\beta + 7 = 0$ ...(i)

Similarly, $L_2(\alpha, \beta) > 0$

$\therefore\ \left|\frac{8\alpha + 6\beta + 11}{10}\right| = 1 \Rightarrow 8\alpha + 6\beta + 1 = 0$ ...(ii)

Solving equations (i) and (ii), we get

$\alpha = \frac{-23}{25}$ and $\beta = \frac{53}{50}$

$\Rightarrow 100(\alpha + \beta) = 100\left(\frac{-23}{25} + \frac{53}{50}\right) = 14$

11. (b) : $\frac{dy}{dx} \propto \frac{-y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{-ky}{x} \Rightarrow \frac{1}{y}dy = \frac{-k}{x}dx$

Integrating on both sides, we get

$\ln|y| = -k\ln|x| + \ln c$

$\Rightarrow \ln y = \ln\left(\frac{c}{x^k}\right) \Rightarrow x^k y = c$

The curve passes through the points (1, 2) and (8, 1).

$\Rightarrow k = \frac{1}{3}$ and $c = 2$

So, the curve is $x^{1/3} \cdot y = 2 \Rightarrow x \cdot y^3 = 8$

Now, $\left|y\left(\frac{1}{8}\right)\right| = 4$

12. (a) : The ellipse meets the line $\frac{x}{7} + \frac{y}{2\sqrt{6}} = 1$ on x-axis.

$\Rightarrow y = 0 \therefore x = 7 = a$

And the ellipse meets the line $\frac{x}{7} - \frac{y}{2\sqrt{6}} = 1$ on y-axis.

$\Rightarrow x = 0$

$\therefore\ y = -2\sqrt{6} \Rightarrow b = 2\sqrt{6}$

$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{24}{49}} = \frac{5}{7}$

13. (d) : The tangents at point $A(1, 3)$ on the parabola $y^2 - 2x - 2y = 1$ is

$y_1 y - (x + x_1) - (y + y_1) - 1 = 0$

$\Rightarrow 3y - (x + 1) - (y + 3) - 1 = 0$

$\Rightarrow 3y - x - 1 - y - 3 - 1 = 0 \Rightarrow 2y - x - 5 = 0$

$\Rightarrow 2y - x = 5$ ...(i)

and for point $B(1, -1)$ is

$-y - (x + 1) - (y - 1) - 1 = 0$

$\Rightarrow -y - x - 1 - y + 1 - 1 = 0 \Rightarrow -2y - x - 1 = 0$

$\Rightarrow 2y + x = -1$ ...(ii)

Solving equations (i) and (ii), we get $x = -3$ and $y = 1$

So, the point of intersection of the tangents P is $(-3, 1)$.

So, area of $\Delta PAB = \frac{1}{2}\left\|\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix}\right\| = \frac{1}{2}\left\|\begin{matrix} 1 & 3 & 1 \\ 1 & -1 & 1 \\ -3 & 1 & 1 \end{matrix}\right\|$

$= \frac{1}{2}\left|1(-1-1) - 3(1+3) + 1(1-3)\right|$

$= \frac{1}{2}\left|-2 - 12 - 2\right|$ = 8 square units

14. (d) : The equation of ellipse is $\frac{x^2}{16} + \frac{y^2}{7} = 1$

$\Rightarrow a_1 = 4, b_1 = \sqrt{7}$

$e_1 = \sqrt{1 - \frac{7}{16}} = \sqrt{\frac{9}{16}} = \frac{3}{4}$ $\left[\because e_1 = \sqrt{1 - \frac{b_1^2}{a_1^2}}\right]$

and the equation of hyperbola is $\frac{x^2}{144} - \frac{y^2}{\alpha} = \frac{1}{25}$

$\Rightarrow \frac{x^2}{\frac{144}{25}} - \frac{y^2}{\frac{\alpha}{25}} = 1 \Rightarrow a_2 = \frac{12}{5}, b_2 = \frac{\sqrt{\alpha}}{5}$

As the foci of the ellipse and the hyperbola coincide

$\therefore\ a_1 e_1 = a_2 e_2$

$\Rightarrow 4 \cdot \frac{3}{4} = \frac{12}{5} e_2 \Rightarrow \frac{5}{4} = e_2$

$e_2 = \sqrt{1 + \left(\frac{b_2}{a_2}\right)^2} \Rightarrow \frac{5}{4} = \sqrt{1 + \left(\frac{\frac{\alpha}{25}}{\frac{144}{25}}\right)}$

$\Rightarrow \frac{25}{16} = 1 + \frac{\alpha}{144} \Rightarrow \alpha = \frac{9}{16} \times 144 \Rightarrow \alpha = 81 \Rightarrow b_2 = \frac{9}{5}$

The length of latus rectum of hyperbola = $\frac{2\left(\frac{81}{25}\right)}{\frac{12}{5}} = \frac{81}{30} = \frac{27}{10}$

15. (c) : The equation of plane E is

$a(x - 1) + b(y + 1) + c(z - 1) = 0$ as it passes through $P(1, -1, 1)$.

As E is perpendicular to planes $2x - 2y + z = 0$

$\Rightarrow 2a - 2b + c = 0$

and also E is perpendicular to plane $x - y + 2z = 4$

$a - b + 2c = 0$

By cross product, $\frac{a}{-4+1} = \frac{b}{1-4} = \frac{c}{0} = \lambda$

$\Rightarrow a = -3\lambda, b = -3\lambda, c = 0$

$\Rightarrow -3\lambda x + 3\lambda - 3\lambda y - 3\lambda = 0 \Rightarrow x + y = 0$

The distance of plane E from point $Q(a, a, 2)$ is $3\sqrt{2}$.

$\Rightarrow \left|\frac{2a}{\sqrt{2}}\right| = 3\sqrt{2} \Rightarrow a = 3, -3$

$\therefore\ Q(3, 3, 2)$ or $Q(-3, -3, 2)$

Now, $PQ = \sqrt{(3-1)^2 + (3+1)^2 + (2-1)^2}$

or $\sqrt{(-3-1)^2 + (-3+1)^2 + (2-1)^2}$

$\Rightarrow PQ = \sqrt{2^2 + 4^2 + 1^2} \Rightarrow PQ^2 = 4 + 16 + 1 = 21$

16. (a) : The given lines can be represented as

$$\frac{x+7}{-6}=\frac{y-6}{7}=z \text{ and } \frac{x-7}{-2}=y-2=z-6$$

$$\text{Shortest distance } (d) = \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(a_1b_2-a_2b_1)^2+(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2}}$$

$$=\frac{\begin{vmatrix} 14 & -4 & 6 \\ -6 & 7 & 1 \\ -2 & 1 & 1 \end{vmatrix}}{\sqrt{8^2+6^2+4^2}}=\frac{116}{\sqrt{116}}=\sqrt{116}=2\sqrt{29}$$

17. (a) : Given, $\vec{a}=\hat{i}-\hat{j}+2\hat{k}$, $\vec{a}\times\vec{b}=2\hat{i}-\hat{k}$ and $\vec{a}\cdot\vec{b}=3$

Projection of $\vec{b}$ on $(\vec{a}-\vec{b})$ is $=\dfrac{\vec{b}\cdot(\vec{a}-\vec{b})}{|\vec{a}-\vec{b}|}$

Now, $|\vec{a}\times\vec{b}|^2+(\vec{a}\cdot\vec{b})^2=|\vec{a}|^2|\vec{b}|^2$

$\Rightarrow\ 5+9=6|\vec{b}|^2 \Rightarrow \dfrac{7}{3}=|\vec{b}|^2 \Rightarrow |\vec{b}|=\sqrt{\dfrac{7}{3}}$

$|\vec{a}-\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2-2(\vec{a}\cdot\vec{b})=6+\dfrac{7}{3}-2\times3=\dfrac{7}{3}$

$\Rightarrow\ |\vec{a}-\vec{b}|=\sqrt{\dfrac{7}{3}}$

Projection $\vec{b}$ on $(\vec{a}-\vec{b})=\dfrac{\vec{b}\cdot\vec{a}-|\vec{b}|^2}{|\vec{a}-\vec{b}|}=\dfrac{3-\frac{7}{3}}{\sqrt{\frac{7}{3}}}=\dfrac{2}{3}\times\sqrt{\dfrac{3}{7}}=\dfrac{2}{\sqrt{21}}$

18. (d) : Numbers are 3, 5, 7, $2k$, 12, 16, 21, 24 and M.D. = 6

Median (M) = $\dfrac{1}{2}[2k+12]$ $(\because\ n=8)$

$=k+6$

Now, $\text{M.D.}=\sum_{i=1}^{8}\dfrac{|x_i-M|}{n}$

$6=\dfrac{1}{8}(k+3+k+1+k-1+6-k+6-k+10-k+15-k+18-k)$

$\Rightarrow$ 48 = 58 – $2k$ $\Rightarrow$ $2k$ = 10 $\Rightarrow$ k = 5 $\therefore$ Median = 5 + 6 = 11

19. (b) : $2\sin\left(\dfrac{\pi}{22}\right)\sin\left(\dfrac{3\pi}{22}\right)\sin\left(\dfrac{5\pi}{22}\right)\sin\left(\dfrac{7\pi}{22}\right)\sin\left(\dfrac{9\pi}{22}\right)$

$=2\cos\left(\dfrac{\pi}{2}-\dfrac{\pi}{22}\right)\cos\left(\dfrac{\pi}{2}-\dfrac{3\pi}{22}\right)\cos\left(\dfrac{\pi}{2}-\dfrac{5\pi}{22}\right)\cos\left(\dfrac{\pi}{2}-\dfrac{7\pi}{22}\right)\cos\left(\dfrac{\pi}{2}-\dfrac{9\pi}{22}\right)$

$=2\cos\left(\dfrac{10\pi}{22}\right)\cos\left(\dfrac{8\pi}{22}\right)\cos\left(\dfrac{6\pi}{22}\right)\cos\left(\dfrac{4\pi}{22}\right)\cos\left(\dfrac{2\pi}{22}\right)$

$=2\cos\left(\dfrac{16\pi}{11}\right)\cos\left(\dfrac{4\pi}{11}\right)\cos\left(\dfrac{8\pi}{11}\right)\cos\left(\dfrac{2\pi}{11}\right)\cos\left(\dfrac{\pi}{11}\right)$

$\left(\because\ \cos\dfrac{5\pi}{11}=-\dfrac{\cos16\pi}{11} \text{ and } \cos\dfrac{3\pi}{11}=-\cos\dfrac{8\pi}{11}\right)$

$=2\cos\dfrac{16\pi}{11}\cos\dfrac{4\pi}{11}\cos\dfrac{8\pi}{11}\cos\dfrac{2\pi}{11}\cos=\dfrac{2\times\sin\left(\frac{32\pi}{11}\right)}{2^5\sin\frac{\pi}{11}}=\dfrac{1}{2^4}=\dfrac{1}{16}\dfrac{\pi}{11}$

$\left[\because \cos A\cos2A\cos4A\ldots\cos2^{n-1}A=\dfrac{\sin(2^nA)}{2^n\sin A}\right]$

20. (d) **21. (107)**

22. (25) : Let $f(x)=x^2+bx+c$

As $f(0)=p \Rightarrow c=p \Rightarrow f(x)=x^2+bx+p$

and $f(1)=\dfrac{1}{3} \Rightarrow \dfrac{1}{3}=1+b+p$ ...(i)

Let α be the common root of $f(x)=0$ and $fofofof(x)=0$

then $f(\alpha)=0$ and $fofof(0)=0 \Rightarrow fof(p)=0$

$\Rightarrow\ f(p^2+bp+p)=0 \Rightarrow p^2+bp+p=\alpha$ or β if $p(p+b+1)=\alpha$

$\Rightarrow\ \dfrac{p}{3}=\alpha$ [From (i)]

By product of roots = p, $\dfrac{p}{3}\times\beta=p \Rightarrow \beta=3\ \therefore\ (\alpha,\beta)=\left(\dfrac{p}{3},3\right)$

$f(\beta)=0 \Rightarrow 9+3b+p=0$ ...(ii)

Solving (i) and (ii), equation, we get

$b=\dfrac{-25}{6}$ and $p=\dfrac{7}{2}$

$\therefore\ f(x)=x^2-\dfrac{25}{6}x+\dfrac{7}{2} \Rightarrow f(-3)=9+\dfrac{25}{2}+\dfrac{7}{2}=25$

23. (24) : $A=\begin{bmatrix}1&a&a\\0&1&b\\0&0&1\end{bmatrix}$; $A^2=\begin{bmatrix}1&a&a\\0&1&b\\0&0&1\end{bmatrix}\begin{bmatrix}1&a&a\\0&1&b\\0&0&1\end{bmatrix}$

$=\begin{bmatrix}1&2a&a+ab+a\\0&1&2b\\0&0&1\end{bmatrix}=\begin{bmatrix}1&2a&2a+ab\\0&1&2b\\0&0&1\end{bmatrix}$

$A^3=\begin{bmatrix}1&a&a\\0&1&b\\0&0&1\end{bmatrix}\begin{bmatrix}1&2a&2a+ab\\0&1&2b\\0&0&1\end{bmatrix}=\begin{bmatrix}1&3a&3a+3ab\\0&1&3b\\0&0&1\end{bmatrix}$

$A^4=\begin{bmatrix}1&a&a\\0&1&b\\0&0&1\end{bmatrix}\begin{bmatrix}1&3a&3a+3ab\\0&1&3b\\0&0&1\end{bmatrix}=\begin{bmatrix}1&4a&4a+6ab\\0&1&4b\\0&0&1\end{bmatrix}$

$\therefore$ We can say $na=48$, $nb=96$, $na+\dfrac{n(n-1)}{2}ab=2160$

By solving these equations, we get

$n=12$, $a=4$ and $b=8 \Rightarrow n+a+b=12+4+8=24$

24. (15) : $f(x)=|5x-7|+[x^2+2x]$

Here, x^2+2x is a strictly increasing function in interval $\left[\dfrac{5}{4},2\right]$.

At $x=5/4$ the function $[x^2+2x]=4$ *i.e.*, the minimum value.

And for $|5x-7|$, the minimum value is obtained for 7/5.

For $f(x)$ the minimum value can be attained at 7/5 *i.e.*, 4.

The maximum value for the function $f(x)$ can be attained at 2.
At $x = 2$, the $f(x) = |5x - 7| + [x^2 + 2x] = 11$
$\therefore$ Maximum value = 11
Minimum value = 4
Sum = 15

25. (3) : The given differential equation is $\frac{dy}{dx} = \frac{4y^3 + 2yx^2}{3xy^2 + x^3}$

It can be represented as $\frac{dy}{dx} = \frac{4 + 2\left(\frac{x}{y}\right)^2}{3\left(\frac{x}{y}\right) + \left(\frac{x}{y}\right)^3}$

Let $v = \frac{y}{x} \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$; $v + x\frac{dv}{dx} = \frac{4 + 2\left(\frac{1}{v}\right)^2}{\frac{3}{v} + \left(\frac{1}{v}\right)^3}$

$\Rightarrow x\frac{dv}{dx} = \frac{4v^3 + 2v - 3v^3 - v}{3v^2 + 1} = \frac{v^3 + v}{3v^2 + 1}$

$\Rightarrow \int\left(\frac{3v^2 + 1}{v^3 + v}\right)dv = \int\frac{dx}{x}$

$\Rightarrow \int\frac{dt}{t} = \ln x + \ln c$ [Let $v^3 + v = t \Rightarrow (3v^2 + 1)dv = dt$]

$\Rightarrow \ln|t| = \ln(x \cdot c) \Rightarrow |v^3 + v| = x \cdot c$

$\Rightarrow \left(\frac{y}{x}\right)^3 + \left(\frac{y}{x}\right) = x \cdot c \Rightarrow \frac{y^3}{x^3} + \frac{y}{x} = x \cdot c \Rightarrow y^3 + yx^2 = x^4 \cdot c$

As $y(1) = 1$
$\Rightarrow 2 = c \Rightarrow y^3 + yx^2 = 2x^4$
As $y(2) \in [n - 1, n)$
$\Rightarrow y^3 + 4y = 32$
Let $f(y) = y^3 + 4y - 32$
$f(3) > 0$ and $f(2) < 0$
$f(2) \cdot f(3) < 0$
$\therefore$ One root lies in [2, 3)
$\therefore$ $n = 3$

26. (8) : $f(x) + \int_0^x (x - t)f'(t)dt = (e^{2x} + e^{-2x})\cos 2x + \frac{2x}{a}$

Differentiating on both sides, we get

$f'(x) + 0 = -2(e^{2x} + e^{-2x})\sin 2x + 2\cos 2x[e^{2x} - e^{-2x}] + \frac{2}{a}$

Put $x = 0$

$\Rightarrow f'(0) = -0 + 2 \times 1 \times 0 + \frac{2}{a} \Rightarrow 4 = \frac{2}{a} \Rightarrow a = \frac{1}{2}$

Now, $(2a + 1)^5 a^2 = (2)^5\left(\frac{1}{2}\right)^2 = 2^3 = 8$

27. (5) : $a_n = \int_{-1}^{n}\left(1 + \frac{x}{2} + \frac{x^2}{3} + ... + \frac{x^{n-1}}{n}\right)dx$

$a_1 = \int_{-1}^{1} 1dx = [x]_{-1}^{1} = 2 \notin (2, 30)$

$a_2 = \int_{-1}^{2}\left(1 + \frac{x}{2}\right)dx = \left[x + \frac{x^2}{4}\right]_{-1}^{2}$

$= (2 + 1) - \left(-1 + \frac{1}{4}\right) = \frac{15}{4}$

$a_3 = \int_{-1}^{3}\left(1 + \frac{x}{2} + \frac{x^2}{3}\right)dx = \left(x + \frac{x^2}{4} + \frac{x^3}{9}\right)_{-1}^{3}$

$= \left(3 + \frac{9}{4} + 3\right) - \left(-1 + \frac{1}{4} - \frac{1}{9}\right) = \frac{33}{4} + \frac{31}{36} = \frac{328}{36} = 9.11$

$a_4 = \int_{-1}^{4}\left(1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4}\right)dx = \left(x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16}\right)_{-1}^{4}$

$= \left(4 + 4 + \frac{64}{9} + 16\right) - \left(-1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16}\right)$

$= 24 + \frac{64}{9} + \frac{115}{144} = \frac{280}{9} + \frac{115}{144}$

$= 31.11 + 0.79 = 31.90 \notin (2, 30)$
So, sum of all elements = 2 + 3 = 5

28. (25) : Given circles are $x^2 + y^2 + 6x + 8y + 16 = 0$ ...(i)
and $x^2 + y^2 + 2(3 - \sqrt{3})x + 2(4 - \sqrt{6})y = k + 6\sqrt{3} + 8\sqrt{6}$ ...(ii)

For (i) circle, $c_1 = (-3, -4)$ and $r_1 = 3$
For (ii) circle, $c_2 = (-3 + \sqrt{3}, -4 + \sqrt{6})$

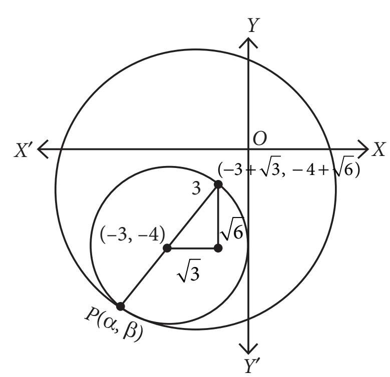
$r_2 = \sqrt{(3 - \sqrt{3})^2 + (4 - \sqrt{6})^2 + k + 6\sqrt{3} + 8\sqrt{6}}$

$= \sqrt{9 + 3 - 6\sqrt{3} + 16 + 6 - 8\sqrt{6} + 6\sqrt{3} + 8\sqrt{6} + k} = \sqrt{34 + k}$

As circles touch internally,
$c_1c_2 = |r_2 - r_1|$ $\left[\because 3 = |\sqrt{34 + k} - 3| \Rightarrow 6 = \sqrt{34 + k}\right]$
$\Rightarrow k = 2$
$\Rightarrow r_2 = \sqrt{34 + 2} = 6$

Distance between the centre of the circle is 3 and $r_1 = 3$

So, $(-3, -4)$ is the mid point of $(-3 + \sqrt{3}, -4 + \sqrt{6})$ and (α, β)



$\Rightarrow -3 = \frac{\alpha - 3 + \sqrt{3}}{2}$

$\Rightarrow -6 + 3 - \sqrt{3} = \alpha$

$\Rightarrow -3 - \sqrt{3} = \alpha$

$\Rightarrow -4 = \frac{\beta - 4 + \sqrt{6}}{2}$

$\Rightarrow -8 + 4 - \sqrt{6} = \beta$

$\Rightarrow -4 - \sqrt{6} = \beta$

Now, $(\alpha + \sqrt{3})^2 + (\beta + \sqrt{6})^2 = 3^2 + 4^2 = 25$

29. (170) : $f(x) = 4x^3 - 3xy^2 + 6x^2 - 5xy - 8y^2 + 9x + 14$
Differentiate w.r.t. x, we get

$\Rightarrow\ 12x^2-3y^2-6xy\frac{dy}{dx}+12x-5x\frac{dy}{dx}-5y-16y\frac{dy}{dx}+9=0$

At point (–2, 3)

$\Rightarrow\ 12\times4-3\times9+36\frac{dy}{dx}-24+10\frac{dy}{dx}-15-48\frac{dy}{dx}+9=0$

$\Rightarrow\ -9-2\frac{dy}{dx}=0\ \Rightarrow\ \frac{dy}{dx}=\frac{-9}{2}$

Equation of tangent at (–2, 3) is

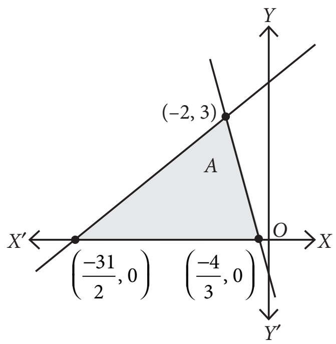
$y-3=m_T(x+2)$

$\Rightarrow\ y-3=\frac{-9}{2}(x+2)$

$\Rightarrow\ 9x+2y+12=0$

Equation of normal is

$2x-9y+31=0$



Area of $A=\frac{1}{2}\times\frac{85}{6}\times3$

$8A=\frac{1}{2}\times\frac{85}{6}\times3\times8=170$

30. (130) : $x=\sin(2\tan^{-1}\alpha)$ and $y=\sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$

$x=\sin(2\theta),\ \theta=\tan^{-1}\alpha$

$\Rightarrow\ x=\frac{2\tan\theta}{1+\tan^2\theta},\ \alpha=\tan\theta\ \Rightarrow\ x=\frac{2\alpha}{1+\alpha^2}$ ...(i)

and $y=\sin\left(\frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right)\right)$

$y=\sin\theta,\ \frac{1}{2}\tan^{-1}\frac{4}{3}=\theta\ \Rightarrow\ y=\sin\theta,\ \tan2\theta=\frac{4}{3}$

$\Rightarrow\ y=\sin\theta,\ \tan\theta=\frac{1}{2}\ \Rightarrow\ y=\frac{1}{\sqrt{5}}$

Now, $y^2=1-x$

$\frac{1}{5}=1-\frac{2\alpha}{1+\alpha^2}\ \Rightarrow\ \frac{1}{5}=\frac{1+\alpha^2-2\alpha}{1+\alpha^2}$

$\Rightarrow\ 2\alpha^2-5\alpha+2=0\ \Rightarrow\ \alpha=\frac{1}{2},\ 2$

$\sum_{\alpha\in S}16\alpha^3=16\times(2)^3+16\times\left(\frac{1}{2}\right)^3=128+2=130$